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Evolution of Grassmannian invariant-angle coherent states and nonadiabatic Hannay's angle

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Abstract. We show how the exact evolution and nonadiabatic Hannay's angle of Grassmannian classical mechanics of spin one half in a varying external magnetic field is associated with the evolution of Grassmannian invariant-angle coherent states.

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Considering the evolution of a quantum system whose Hamiltonian $H(t)$ depends explicitly on time through a set of parameters $\vec{X}(t)$, Berry [1] has shown that, when $\vec{X}(t)$ describes adiabatically a loop $\mathcal{C}$ in parameters space, an eigenstate of the Hamiltonian develops, besides an expected dynamical phase, a geometrical phase $\gamma_n(\mathcal{C})$ which depends essentially on the closed path $\mathcal{C}$ which has been followed in parameters space. Removing the adiabatic hypothesis, Aharonov and Anandan [2] have generalized Berry's result and shown that such a geometrical phase may appear for any state which is cyclic with respect to some evolution. (Cyclicity means that the state returns to itself, after some time up to a phase; in Berry's approach, the adiabatic hypothesis ensures the cyclicity of the eigenstates of $H(t)$ after one loop). Therefore, in the study of quantum nonadiabatic holonomy effects, a complete set of cyclic states play the same basic role as the Hamiltonian eigenvectors in the adiabatic case. A natural (but not unique) way to get such a basis of cyclic states is to consider the eigenvectors of a Hermitian periodic invariant $I(t)$ defined by

$$\frac{\partial I}{\partial t} = \imath\hbar[I, H]\,. \qquad (1)$$

Indeed, any eigenstate $|n, 0\rangle$ (relative to the time-independent eigenvalue λ_n) of an invariant operator $I(0)$ at time zero evolves continuously into the corresponding eigenstate $|n, t\rangle$ of the invariant operator $I(t)$ at time t [3], exactly as an eigenstate of the Hamiltonian does when the evolution is adiabatic. For this reason, invariant theory takes an important place in recent works on nonadiabatic geometric phases [4–9].

The classical analog of Berry's phase is Hannay's angle [10]. According to the classical adiabatic theorem, any trajectory in phase space of the classical integrable Hamiltonian at time zero evolves into a trajectory of the Hamiltonian at time t with the same action. Hannay [10] has shown that when the adiabatic excursion takes place on a closed path in the space of parameters, an extra shift analogous to Berry's phase is realized in the angle variables. This extra angle depends on the geometry of the parameter space circuit and on the conserved actions. It can be viewed as a semiclassical limit of Berry's phase [11]. A geometrical angle can be defined on a constant-action surface for a cyclic evolution [12,13] in a classical nonadiabatic integrable Hamiltonian system; this angle is the classical counterpart of the geometrical phase [2], so it is called the nonadiabatic Hannay's angle.

For a quantum Hamiltonian with integrable classical limit in ordinary phase space, the quantum-classical correspondence is well understood for action-angle coherent states [14]. In the classical limit such a quantum Hamiltonian possesses an infinite number of energy levels. However, also Hamiltonians with a finite number of levels are of interest. A well-studied system is the two-level system of a spin-$(1/2)$ magnetic dipole coupled in a slowly varying external magnetic field. This system can be written as a classical model by means of Grassmann variables [15,16]. The corresponding classical adiabatic holonomy and Hannay's angle were investigated by Gozzi and Thacker [17]. Using fermionic coherent states and canonical transformation, Abe [18] gives an alternative derivation of the result obtained in reference [17]. Hannay's angle was proved to be simply related to Berry's phases relative to suitable action-angle coherent states [19].

The important point which we emphasise in this paper is that the nonadiabatic Hannay's angle of Grassmanian classical mechanics [20] is associated with the evolution of Grassmannian invariant-angle coherent states.

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To be explicit, we start with the exact classical evolution and nonadiabatic Hannay's angle of a Grassmann spin in a time-dependent magnetic field. As shown in reference [17], the Hamiltonian of a Grassmann spin

$$H = -\frac{i}{2}\varepsilon_{kij} B_k(t)\xi_i\xi_j\,, \tag{2}$$

involves 3 real Grassmann variables ξ_i with Grassmann algebra $\xi_i\xi_j + \xi_j\xi_i = 0$. It leads to Pauli spin in the time-dependent magnetic field

$$H = \frac{1}{2}\vec{B}(t)\,\vec{\sigma} \tag{3}$$

when the anticommuting three-vectors $\vec{\xi}$ are transformed to the Pauli matrix after the quantization $\hat{\xi} = \vec{\sigma}/\sqrt{2}$.

It is easy to show that the system described by the Hamiltonian (2) admits a time-dependent invariant

$$\begin{aligned} I(t) = {}& -\frac{i}{2}\varepsilon_{kij}R_k(t)\xi_i\xi_j = \\ & -\frac{i}{4}\Bigg\{(r^2 + r^{*2})\varepsilon_{1lm} - i(r^2 - r^{*2})\varepsilon_{2lm} \\ & + \frac{2}{B_+}\left(ir\dot{r} + \frac{B_3 r^2}{2}\right)\varepsilon_{3lm}\Bigg\}\xi_l\xi_m \end{aligned} \tag{4}$$

satisfying the relation

$$\frac{\partial I}{\partial t} = -\{H(\vec{\xi}), I(\vec{\xi})\}_{\mathrm{PB}} \equiv -iH(\vec{\xi})\overleftarrow{\partial_j}\,.\,\overrightarrow{\partial_j} I(\vec{\xi})\,, \tag{5}$$

where $\overleftarrow{\partial}_j$ and $\overrightarrow{\partial}_j$ are right and left derivatives with respect to ξ_j, $r(t)$ is the solution of the following auxiliary equation:

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\dot{r}}{B_+}\right) + \frac{r}{4}\left[\frac{B_- B_+ + {B_3}^2}{B_+} - 2i\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{B_3}{B_+}\right)\right] - \frac{B_+}{r^3} = 0, \tag{6}$$

$B_\pm = B_1 \pm iB_2$ and r^* denotes the complex conjugate of r.

When expressed in terms of its normal modes, the invariant $I(t)$ takes the form

$$I = -\frac{1}{2}\zeta_1^*\zeta_1 + \frac{1}{2}\zeta_2^*\zeta_2 = -\zeta_1^*\zeta_1 = \zeta_2^*\zeta_2\,, \tag{7}$$

where ζ_1 and ζ_2 are complex conjugates of each other: $\zeta_2 = \zeta_1^*$, while $\zeta_3 = \zeta_3^*$ is real. The complex normal coordinates ζ_i are deduced from ξ_j's through the unitary transformation $\zeta_i = (S^+)_{ij}\xi_j$ $(i,j = 1,2,3)$ which diagonalizes the invariant $I(t)$. The angle variables of this classical system are $\theta_a = -\mathrm{Arg}\zeta_a$ $(a = 1,2)$ and are determined by Cherbal [20] using a classical time-dependent canonical transformation. The geometrical part or nonadiabatic Hannay's angle appears in the angle variables $\theta_a(t)$ after a cyclic evolution.

Since we shall be interested in this paper by a purely quantum interpretation of an exact solution and a nonadiabatic geometrical angle associated with Grassmannian classical spin, let us recall how these angles appear in the action-angle coherent states when dealing with ordinary (commutative) classical mechanics. The action-angle coherent states are defined in the classical approximation, that is for $\hbar$ small with respect to the classical action, in a way which resembles the definition of the usual (harmonic oscillator) coherent states:

$$|\alpha, \vec{X}(t)\rangle = e^{-|\alpha|^2/2}\sum_{n=0}^{\infty}\frac{\alpha^n}{\sqrt{n!}}|n, \vec{X}(t)\rangle\,, \tag{8}$$

where $|n, \vec{X}(t)\rangle$ are eingenstates of the Hamiltonian $H(\vec{X}(t))$ which depend on parameters $\vec{X}(t)$ slowly varying in time. We call them "action-angle" coherent states because the complex number α can be related to the classical action-angle variables by $\alpha = \sqrt{I/\hbar}\,e^{-i\theta}$. Indeed, when the parameters $\vec{X}$ are fixed, the quantum evolution of $|\alpha, \vec{X}\rangle$ amounts up to a global unessential phase factor, to keep the modulus of α constant and to change θ into $\theta + \frac{\partial E_N}{\partial N}t$. This allows the identification of θ with the classical angle variable. Moreover, in the classical limit ($\hbar$ goes to zero, $|\alpha|$ goes to infinity with the product $|\alpha|^2\hbar$ remaining finite) the sum (8) over n is peaked around $N = |\alpha|^2$ and the relation $I = |\alpha|^2\hbar$ is nothing but the correspondence principle. When the parameters vary slowly with time each eigenfunction $|n, \vec{X}(t)\rangle$ acquires the extra phase $\gamma_N^B(t)$ inducing a change of the coherent state such that the modulus of α remains constant while its argument θ becomes $\theta - \frac{\partial\gamma_N^B}{\partial N}(t)$. Then $\theta_I^{\mathrm{H}}(t) = -\frac{\partial\gamma_N^B}{\partial N}(t)$ defines the corresponding Hannay's angle in classical mechanics. We have exemplified the quantum-classical correspondence at the level of action-angle coherent states. Let us note that the mean value of the quantum Hamiltonian in these states

$$\langle\alpha, \vec{X}(t)|H(\vec{X}(t))|\alpha, \vec{X}(t)\rangle = H_{\mathrm{c}}(I, \vec{X}(t)) \tag{9}$$

can be identified with the classical Hamiltonian H_{c} (which is a function of the action only).

Let us now present the Grassmannian invariant-angle coherent-state approach of this model. We shall find suitable Grassmannian (or fermionic) invariant-angle coherent states $|\zeta, t\rangle$ which have the property of the state $|\alpha, \vec{X}(t)\rangle$: that every change in the phase of quantum invariant's eigenstates $|n,t\rangle \to e^{i\phi_n}|n,t\rangle$ induces a change $\zeta \to \zeta e^{i\theta}$ of the arguments of the parameter of Grassmannian invariant-angle coherent states, and the classical fermionic invariants are precisely the expectation value of the corresponding quantum invariants. The difference with the commutative case is that now there is no need of a classical limit $n \to \infty$ and $\hbar \to 0$. (We therefore set $\hbar = 1$ in the following.)

Let us express the quantum invariant $I = \vec{R}(t)\,\vec{\sigma}/2$ corresponding to the classical one (4) in terms of fermionic operators $b(t)$ which annihilate the lowest eigenstate $|0,t\rangle$ of I and $b^+(t)$ which brings this state onto the other eigenstate $|1,t\rangle$ as

$$I(t) = \left(b^+(t)\,b(t)\, -\, 1/2\right). \tag{10}$$

$$\begin{pmatrix} b(t) \\ b^+(t) \\ c(t) \end{pmatrix} = U^+ \begin{pmatrix} b \\ b^+ \\ c \end{pmatrix} = \frac{1}{2} \begin{pmatrix} (1+R_3)r/r^* & -(1-R_3)r^*/r & -|r|^2/\sqrt{2} \\ -(1-R_3)r/r^* & (1+R_3)r^*/r & -|r|^2/\sqrt{2} \\ r^2/\sqrt{2} & r^{*2}/\sqrt{2} & 2R_3 \end{pmatrix} \begin{pmatrix} b \\ b^+ \\ c \end{pmatrix} , \tag{11}$$

$$\theta^{\mathrm{D}} = \int_0^t \mathrm{d}t' \langle 1,t'|\frac{1}{2}(B_+(t'), B_-(t'), \sqrt{2}B_3(t'))\, U(t') \begin{pmatrix} b(t') \\ b^+(t') \\ c(t') \end{pmatrix} |1,t'\rangle$$
$$- \int_0^t \mathrm{d}t' \langle 0,t'|\frac{1}{2}(B_+(t'), B_-(t'), \sqrt{2}B_3(t'))\, U(t') \begin{pmatrix} b(t') \\ b^+(t') \\ c(t') \end{pmatrix} |0,t'\rangle ; \tag{16}$$

The time-dependent fermionic operators $b(t)$ and $b^+(t)$ are related to the operators $\widehat{\xi}_i$ via the time-dependent unitary transformation U

see equation (11) above

where the operators $b = (\widehat{\xi}_1 - i\widehat{\xi}_2)/\sqrt{2}$, $b^+ = (\widehat{\xi}_1 + i\widehat{\xi}_2)/\sqrt{2}$ and $c = c^+ = \widehat{\xi}_3$ satisfy the algebra

$$\{b, b^+\}_+ = \{c, c\}_+ = 1 ,$$
$$\{b, b\}_+ = \{b, c\}_+ = 0 . \tag{12}$$

(In the matrix notation, $b^+ = \sigma_+$ and $b = \sigma_-$ with $\sigma_\pm = (\sigma_1 \pm i\sigma_2)/2$, and the Clifford number c is $\sigma_3/\sqrt{2}$.) Therefore, the time-dependent operators $b(t)$, $b^+(t)$ and $c(t)$ obviously satisfy the algebra isomorphic to equation (12). The initial Grassmannian invariant-angle coherent states are taken to be

$$|\zeta(0), 0\rangle = \exp\left[-\frac{1}{2}\zeta^*(0)\zeta(0)\right]\big(|0,0\rangle - \zeta(0)|1,0\rangle\big) \tag{13}$$

and are eigenstates of $b(0)$ with eigenvalue $\zeta(0)$, and they are created from the ground state $|0,0\rangle$ by the unitary operator $\exp\left(-[\zeta(0)b^+(0) + \zeta^*(0)b(0)]\right)|0,0\rangle$. According to the Lewis-Riesenfeld theory, one can immediately see that the evolution

$$|0,0\rangle \to e^{i\phi_0(t)}\,|0,t\rangle$$

and

$$|1,0\rangle \to e^{i\phi_1(t)}\,|1,t\rangle \tag{14}$$

of the eigenstates of $I(0)$ induces the evolution of Grassmannian invariant-angle coherent states

$$|\zeta(0), 0\rangle \to e^{i\phi_0(t)}\,|\zeta(0)\, e^{i\{\phi_1(t)-\phi_0(t)\}}, t\rangle = |\zeta(t), t\rangle \tag{15}$$

i.e. the argument of parameter ζ changes in the evolution. As is well known, the global phases $\phi_n(t)$ $(n = 0, 1)$ contain a dynamical part $\phi_n^{\mathrm{D}} = -\int_0^t \langle n,t'|H(t')|n,t'\rangle \mathrm{d}t'$ and a geometrical one $\phi_n^{\mathrm{G}} = i\int_0^t \langle n,t'|\partial/\partial t'|n,t\rangle \mathrm{d}t'$. The main point of this elementary result is that the argument $\phi_1(t) - \phi_0(t)$ of the parameter $\zeta(t)$ contain a dynamical part $\phi_1^{\mathrm{D}} - \phi_0^{\mathrm{D}}$ and a geometrical part $\phi_1^{\mathrm{G}} - \phi_0^{\mathrm{G}}$. This geometrical part is nothing but (minus) Hannay's angle [10] in a cyclic evolution. The second key property $I_{\mathrm{c}} = \langle\zeta(t),t|I(t)|\zeta(t),t\rangle + 1/2 = \zeta^*(0)\zeta(0)$ is an immediate consequence of (10) and (15). It allows the identification of the ζ's entering into the definition of $|\zeta,t\rangle$ with the classical normal modes and justifies the Grassmannian invariant-angle coherent-states denomination of $|\zeta,t\rangle$: $\zeta^*\zeta$ is the classical invariant variable.

Let us embark on the calculation of these angles. From equations (3) and (11), we have

see equation (16) above

we see that only the term proportional to $c(t)$ contributes to the calculation of the dynamical angle, which yields

$$\theta^{\mathrm{D}} = \int_0^t \vec{R}\,(t') \cdot \vec{B}\,(t')\mathrm{d}t' . \tag{17}$$

Using equation (11), the $\partial b^+/\partial t$ can be expressed as

$$\frac{\partial b^+}{\partial t} = R_3\left(\frac{\dot{r}^*}{r^*} - \frac{\dot{r}}{r}\right) b^+(t)$$
$$+ \left(\frac{\dot{R}_3}{\sqrt{2}|r|^2} + \frac{1}{4\sqrt{2}}\big(r\dot{r}^* - r^*\dot{r}\big)\right) c(t) , \tag{18}$$

so that

$$\theta^{\mathrm{G}} = -i\int_0^t \mathrm{d}t' \left(\langle 1,t'|\frac{\partial}{\partial t'}|1,t\rangle - \langle 0,t'|\frac{\partial}{\partial t'}|0,t\rangle\right) =$$
$$-i\int_0^t \mathrm{d}t' \langle 1,t'|\frac{\partial b^+}{\partial t'}|0,t\rangle = -i\int_0^t \mathrm{d}t' R_3\left(\frac{\dot{r}^*}{r^*} - \frac{\dot{r}}{r}\right) . \tag{19}$$

For a cyclic evolution of duration T the nonadiabatic Hannay's angle is

$$\theta^{\mathrm{G}} = -i\oint_{\mathcal{C}} R_3\left(\frac{\mathrm{d}r^*}{r^*} - \frac{\mathrm{d}r}{r}\right) . \tag{20}$$

We note here that r must return to its original value, and indeed there do exist such solutions to equation (6). These above results agree with those obtained by Cherbal [20] in the classical Grassmannian case and by Maamache [21] for the classical bosonic model of spin one half.

In the example we studied there is only one angle variable $\theta_1(t) = \phi_1(t) - \phi_0(t)$ instead of two corresponding to the classical normal modes. The reason is that the fundamental state $|0, t\rangle$ taken to be the "vacuum" is not left invariant by the evolution. (We note that the second normal mode could be obtained if $|1, t\rangle$ has been chosen as "vacuum". This is done by interchanging the roles of $b(t)$ and $b^+(t)$; in this case we would obtain the second angle $\theta_0 = -\theta_1$.) Although this approach has the attractive consequence that Hannay's angle appears as the difference of two Berry's phases, it has the major drawback of privileging one of the two eigenstates and thus not allowing a generalisation to the case $N > 2$ levels.

References

1. M.V. Berry, Proc. R. Soc. Lond. A **392**, 45 (1984); see reference [11].
2. Y. Aharonov, J. Anandan, Phys. Rev. Lett. **58**, 1593 (1987).
3. H.R. Lewis, W.B. Riesenfeld, J. Math. Phys. **10**, 1458 (1969).
4. D.A. Molares, J. Phys. A **21**, L889 (1988).
5. M. Cervero, J.D. Lejarreta, J. Phys. A **22**, L1663 (1989).
6. X. Gao, J.B. Xu, T.Z. Qian, Ann. Phys. (N.Y) **204**, 235 (1990).
7. S.K. Bose, B. Dutta Roy, Phys. Rev. A **43**, 3217 (1991).
8. M. Maamache, Phys. Rev. A **52**, 936 (1995), J. Phys. A **29**, 2833 (1996); Ann. Phys. (N.Y.) **254**, 1 (1997).
9. J.Y. Ji, J.K. Kim, S.P. Kim, K.S. Soh, Phys. Rev. A **52**, 3352 (1995).
10. J.H. Hannay, J. Phys. A **18**, 221 (1985).
11. M.V. Berry, J. Phys. A **18**, 15 (1985).
12. M.V. Berry, J.H. Hannay, J. Phys. A **21**, L325 (1988).
13. A. Bhattacharjee, T. Sen, Phys. Rev. A **38**, 6389 (1988).
14. M. Maamache, J.P. Provost, G. Vallée, J. Phys. A **23**, 5765 (1990); Eur. J. Phys. **15**, 121 (1994).
15. G. Giavarini, E. Gozzi, D. Rohrlich, W.D. Thacker, Phys. Rev. D **39**, 3007 (1989).
16. R. Casalbuoni, Nuovo Cimento A **33**, 115 (1976); **33**, 389 (1976); F.A. Berezin, M.S. Marinov, Ann. Phys. (N.Y) **104**, 336 (1977).
17. E. Gozzi, W.D. Thacker, Phys. Rev. D **35**, 2388 (1987).
18. S. Abe, Phys. Rev. D **39**, 2327 (1989).
19. M. Maamache, J.P. Provost, G. Vallée, Phys. Rev. D **46**, 873 (1992).
20. O. Cherbal, Magister Thesis, Université de Sétif (1998) unpublished.
21. M. Maamache, Physica Scr. **54**, 21 (1996).